

Inductance Calculation Techniques --- Part I: Classical Methods

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INDEX OF SYMBOLS

B	Magnetic flux density (Tesla)
c	Speed of light $\approx 3 \times 10^8$ m/s
C_o	Capacitance per unit length (F/m)
E_m	Magnetic energy storage (Joules)
H	Magnetic field intensity (A/m)
I	Current (A)
J	Current density (A/m ²)
K	Surface current density (A/m)
L_o	Inductance per unit length (H/m)
L	Inductance (H)
N	Coil turns
\mathfrak{R}	Reluctance (A-turns/Wb)
Φ	Flux (Weber)
λ	Flux linkage (Weber-turns)
ϵ_o	Permittivity of free space = 8.854×10^{-12} F/m
μ_o	Magnetic permeability of free space $4\pi \times 10^{-7}$ H/m

1. INTRODUCTION

In the first part of this two part series on inductance calculation techniques, classical methods are developed for solving for the inductance of structures in closed-form. The "magnetoquasistatic" limit of Maxwell's equations are described, boundary conditions are shown and other useful tools such as use of the speed of light and magnetic circuit analogies are explained. Using these techniques, the inductance of simple magnetic structures can often be approximated, and several practical applications of the techniques are shown. In *Part II* of this series, methods are shown for calculating inductance of structures where a closed-form solution is not easily obtained.

2. REVIEW OF MQS LAWS AND BOUNDARY CONDITIONS

1. MQS Laws

Maxwell's equations couple electric fields to magnetic fields, and explain how electromagnetic waves are created. There are four Maxwell's equations, but in magnetic design we only need three: Ampere's Law, Gauss' Magnetic Law and Faraday's Law.

In the magnetoquasistatic regime (MQS), we are concerned with Ampere's Law and Gauss' law², which determine the magnetic field and magnetic flux density. In the MQS region of operation, magnetic energy storage is dominant (as compared to energy

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² A third law, Faraday's law, described how eddy currents are created. Since we're only concerned with low-frequency inductance here, we won't worry about the effects of eddy currents.

stored in the electric field) and the operating frequency is low enough so that wave phenomena are small enough to be ignored [1, pp. 70]. The MQS limit of Ampere's law shows that a flowing current creates a magnetic field, (Figure 1a), or:

$$\oint_C \vec{H} \cdot d\vec{l} \approx \int_S \vec{J} \cdot d\vec{A} \quad [1]$$

where \vec{H} is the magnetic field (Amps/meter) and \vec{J} is current density (Amps/meter²).

Gauss' Magnetic Law says that the flux density integrated over any closed surface equals zero, or:

$$\oint_S \vec{B} \cdot d\vec{A} = 0 \quad [2]$$

Faraday's law shows the mechanisms by which a changing magnetic flux generates eddy currents in a conducting material. The relationship in a conducting (or "Ohmic") material relating the current density and electric field is $\vec{J} = \sigma\vec{E}$ and we can derive (Figure 1b):

$$\frac{1}{\sigma} \oint_C \vec{J} \cdot d\vec{l} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{A} \quad [3]$$

The term on the right of this equation is the negative of the time rate of change of the magnetic flux passing through the surface. This is how induced currents are created in conductors.



Figure 1. Figures illustrating use of Maxwell's equations. (a) Ampere's law. (b) Faraday's law.

The use of these Maxwell's equations are best illustrated by example, as shown in a later section. Before we get there, we need to consider the topic of boundary conditions.

2. Boundary Conditions

Often, application of simple boundary conditions eases the computational complexity of magnetic structures. For instance, consider the case of a thin current sheet, as shown in **Figure 2** where the current flow is out of the page. If we assume that the thickness of the sheet Δ is negligible, use of Ampere's law around the closed contour as shown results in:

$$(H_i - H_o)\ell = I \quad [4]$$

where I is the current enclosed by the contour, H_i is the field inside and H_o is the field outside the current winding. (Note that we ignore the Δ portion of the contour as we

assume it is of infinitesimal thickness and hence the integral over the thickness is zero). We can model the winding as a current sheet, of value $K = I/l$ Amps/meter, resulting in the boundary condition for a thin current sheet:

$$H_i - H_o = K \quad [5]$$

This boundary condition tells us how much the magnetic field H increases from one side of the current sheet to another. This boundary condition is of particular use in calculating the field inside a solenoid (more on this later).

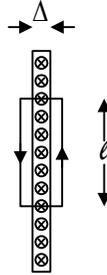


Figure 2. Boundary condition for current sheet

Another important boundary condition can be derived by considering Gauss' magnetic law. This law says that the magnetic flux leaving any closed surface sums to zero. A simple model illustrating this boundary condition is shown in **Figure 3**. What this drawing attempts to illustrate are flux-carrying "pipes", made of high- μ material, with areas A and flux densities B with the directions shown.

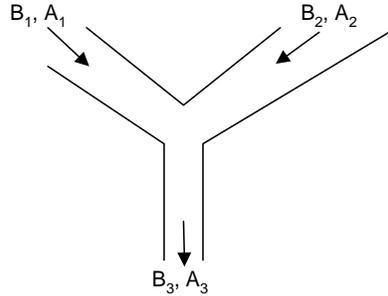


Figure 3. Boundary condition for Gauss' magnetic law

Given these definitions of the direction of the magnetic flux density B , and assuming that B_1 , B_2 and B_3 are constant spatially, Gauss' law results in:

$$B_3 A_3 = B_1 A_1 + B_2 A_2 \quad [6]$$

Use of these boundary conditions can often greatly simplify calculation of some magnetic structures.

3. ANALYTIC TOOLS FOR SOLVING INDUCTANCE PROBLEMS

1. The "Brute Force" Method

The brute force method is the easiest to understand, and often the most difficult to implement. The procedure is as follows:

- Calculate the magnetic flux density B everywhere
- Use this value to calculate the flux Φ
- Once the flux is known, multiply by N to get flux linkage $\lambda = N\Phi$
- The inductance is the flux linkage divided by the coil current, or $L = \lambda/I$.

There are several other indirect methods to calculate the inductance.

2. The Energy Method

Everyone knows the lumped-circuit result for energy stored in an inductor:

$$E_m = \frac{1}{2} LI^2 \quad [7]$$

In many structures, the magnetic field over all space is easily found and the energy stored in the magnetic field can be directly calculated. The energy is found indirectly by integrating the magnetic flux density over all volume, as:

$$E_m = \frac{1}{2\mu_o} \iiint B^2 dV \quad [8]$$

Once the magnetic stored energy is known, it's easy to find the inductance. We apply this method in a later section to a solenoid and to the microstrip line.

3. The Speed of Light

Another technique that is potentially useful is using the speed of light. If the capacitance per unit length (C_o) of your structure is known, the inductance per unit length (L_o) is easily found by using the relationship [5, pp. 459]:

$$c^2 = \frac{1}{L_o C_o} \quad [9]$$

where c is the speed of light. Furthermore, the speed of light in a medium is given by [Haus, pp. 69]:

$$c = \frac{1}{\sqrt{\mu\varepsilon}} \quad [10]$$

where μ and ε are the magnetic permeability and dielectric permittivity of the material, respectively. This technique is useful, as the capacitance of many common structures is known, and by knowing the capacitance by extension you can easily find the inductance.

4. Magnetic Circuit Analogies

The structure of the MQS laws suggests a possible electrical circuit analogy [2], [5, pp. 409]. In a magnetic circuit, flux (Φ) is forced to flow by the total Ampere-turns (NI) driving the circuit. Using this analogy between electrical and magnetic circuits, we can write:

$$V \Leftrightarrow NI \quad [11]$$

$$I \Leftrightarrow \Phi$$

$$R \Leftrightarrow \mathfrak{R}$$

In this case, current is proportional to flux, with the proportionality constant being the reluctance of the magnetic. The analogous relationship between electrical resistance and reluctance is shown below:

$$R = \frac{l}{\sigma A} \Leftrightarrow \mathfrak{R} = \frac{l}{\mu A} \quad [12]$$

The magnetic circuit method is particularly useful for gapped magnetic circuits, and for circuits with multiple paths (where the simple analogy to resistances in parallel is easily seen). The use of magnetic circuit analogies is shown in the next section, where it's applied to a toroid and a C-core magnet.

4. METHODS APPLIED TO COMMON MAGNETIC STRUCTURES

The key to solving for the inductance of magnetic structures is to recognize which of the tools to use: the "brute force" method using Ampere's law, energy methods, the speed of light, or magnetic circuit analogies. If these techniques aren't useful, other computational methods (such as the Biot-Savart law, use of the magnetic vector potential [5] or the use of finite-element analysis) may be employed. Other structures lend themselves to handbook techniques, which is the subject of Part II of this article.

1. Toroid

It's easy enough to find the inductance of a toroid (**Figure 4**) by using the "brute force" method and directly applying Ampere's law and finding the resulting flux in the core. First, we assume that the flux density is uniform inside the core and circulates around the coil axis. Second, we assume that the high- μ material guides all the flux, and that there is no leakage. Using Ampere's law, we can find the flux density in the core by:

$$B_\phi = \mu_c \frac{NI}{\ell_p} \quad [13]$$

The total flux in the core is:

$$\Phi = B_\phi A_c = \mu_c NI \frac{A_c}{\ell_p} \quad [14]$$

and total flux linkage is:

$$\lambda = N\Phi = \mu_c N^2 I \frac{A_c}{\ell_p} \quad [15]$$

The inductance is the flux linkage divided by the applied current, or:

$$L = \frac{\lambda}{I} = \mu_c N^2 \frac{A_c}{\ell_p} \quad [16]$$

Toroids are often used in low cost, off-the-shelf power inductors. For example, the inductance of a toroidal core with $\mu_c = 1000 \mu_0$, $N = 100$, $A_c = 0.25 \text{ cm}^2$ and $\ell_p = 5 \text{ cm}$ is 6.3 milliHenries.

The same result can be derived by using a circuit model. The reluctance of the core is given by:

$$\mathfrak{R} = \frac{\ell_p}{\mu_c A_c} \quad [17]$$

The resultant flux in the core is:

$$\Phi = \frac{NI}{\mathfrak{R}} = \mu_c NI \frac{A_c}{\ell_p} \quad [18]$$

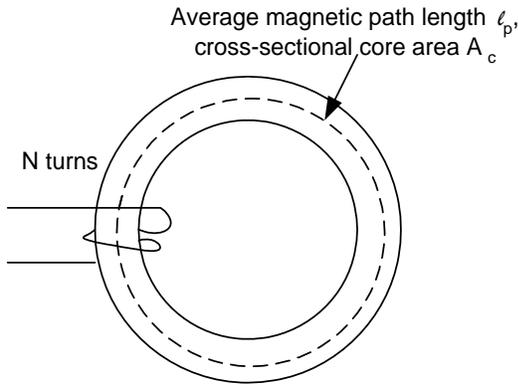


Figure 4. Toroid

2. C-Core with Gap

The gapped-core inductor (**Figure 5**) is also a common structure for power electronics circuits. Using the circuit analogy (**Figure 5b**), the flux in the core is easily found by:

$$\Phi = \frac{NI}{\mathfrak{R}_{core} + \mathfrak{R}_{gap}} = \frac{NI}{\frac{l_p}{\mu_c A_c} + \frac{g}{\mu_o A_c}} \quad [19]$$

Now, note what happens if $g/\mu_o \gg l_p/\mu_c$: The flux in the core is now approximately independent of the core permeability, as:

$$\Phi \approx \frac{NI}{\mathfrak{R}_{gap}} \approx \frac{NI}{\frac{g}{\mu_o A_c}} \quad [20]$$

This is the main reason why it's often useful to use a gapped core: permeability varies with flux level and temperature, and if you don't want your inductance to change you use a gap. The resultant inductance is approximately independent of core properties, as:

$$L = \frac{N\Phi}{I} \approx \frac{N^2}{\mathfrak{R}_{gap}} \approx \frac{N^2}{\frac{g}{\mu_o A_c}} \quad [21]$$

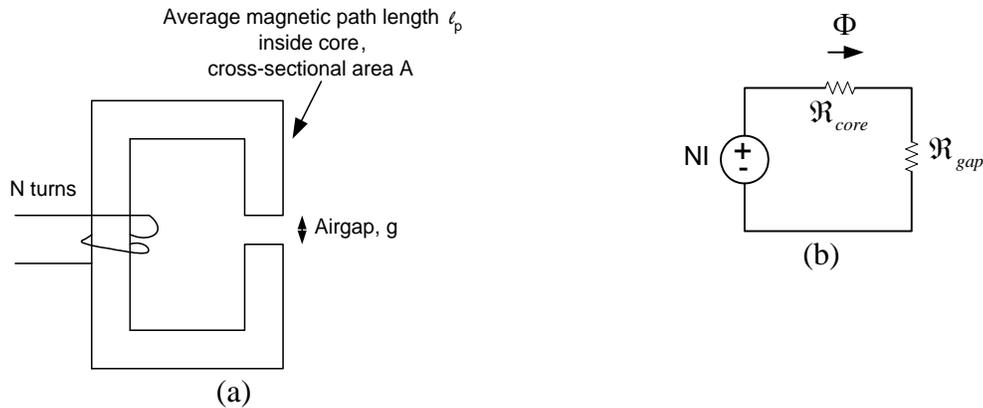


Figure 5. C-core with gap. (a) Structure. (b) Magnetic circuit analogy

3. "Infinitely long" solenoid

We can solve for the inductance of a solenoid of length l (**Figure 6**) if the radius a is much smaller than the length by using assumptions based on an infinitely long solenoid. For an infinitely-long solenoid the magnetic field inside is uniform and axially-directed. If the winding is thin compared to the radius of the solenoid, we can approximate the winding by a current sheet (of value Amps/meter), where the current density is:

$$K = \frac{NI}{l} \quad [22]$$

The field outside the solenoid is ideally zero, and by the MQS boundary condition the H field inside the solenoid equals K . The magnetic flux density B inside is:

$$B = \mu_o \frac{NI}{l} \quad [23]$$

The energy stored in the inductor can be used to find the inductance by:

$$E_m = \frac{1}{2} \frac{B^2}{\mu_o} V = \frac{1}{2} LI^2 \quad [24]$$

where V is the volume of the solenoid bore, resulting in:

$$L = \mu_o N^2 \frac{A}{l} \quad [25]$$

In this case, $A = \pi a^2$, the area of the solenoid bore. Note that this is the same answer obtained if we use a magnetic circuit analogy with reluctance $l/\mu_o A$. Therefore, we can guess that the inductance of a finite length solenoid, with the ends of the solenoid connected together with a high- μ path will have the same inductance as shown above. In *Part II* of this series, we find the inductance of a finite length solenoid.

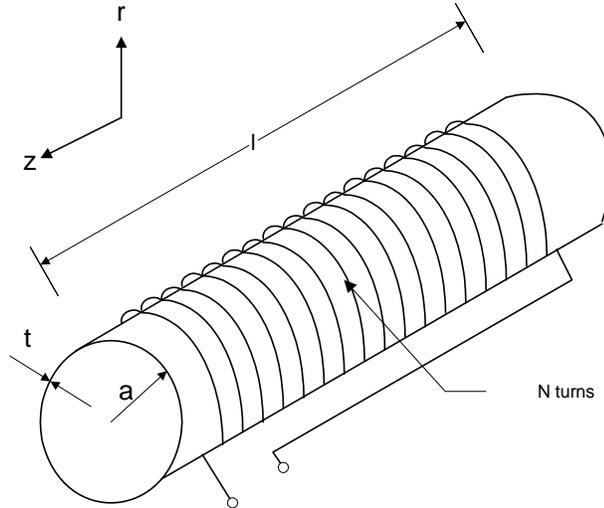


Figure 6. Solenoid

4. *Microstrip Line*

The microstrip line (**Figure 7**) is amenable to using the speed of light relationship, as the capacitance per unit length of a parallel-plate capacitor is well known:

$$C_o = \frac{\epsilon w}{d} \quad [26]$$

where ϵ is the dielectric permittivity of the material inside the capacitor plates. The resulting inductance per unit length is:

$$L_o = \mu \frac{d}{w} \quad [27]$$

Therefore, the inductance of the microstrip line is approximately:

$$L_o = \mu \frac{dl}{w} \quad [28]$$

This is a pretty good approximation if fringing is negligible (or if $d \ll w$). If more accuracy is needed, closed-form solutions based on conformal mapping or other computational techniques can be used [3].

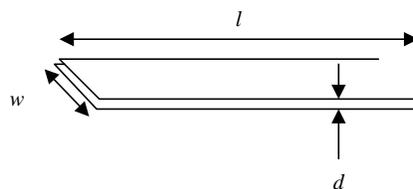


Figure 7. Microstrip line

The microstrip is a useful structure for low-inductance connections on a PC board (for instance, for MOSFET gate drive traces). We can use this relationship to find the inductance of a PC board trace above a ground plane. For instance, a PC board trace 1 centimeter long, 0.5 centimeter wide and 0.04 centimeters above an unbroken ground plane has an inductance of approximately 1 nanoHenry. This inductance can be made arbitrarily small by increasing the trace width (with a resultant increase in the lumped capacitance, of course). This microstrip technique also has application in forming low

inductance paths for high speed current switching, in such applications as semiconductor diode laser driving [4].

5. Coaxial line

For the coaxial line (**Figure 8**), the capacitance per unit length is [5, pp. 177]:

$$C_o = \frac{2\pi\epsilon_o}{\ln(b/a)} \quad [29]$$

Using the speed-of-light relationship, the resulting inductance per unit length is:

$$L_o = \frac{\mu}{2\pi} \ln(b/a) \quad [30]$$

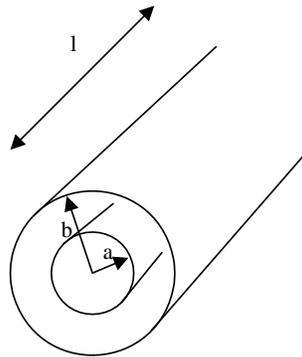


Figure 8. Coaxial line

5. CONCLUSIONS

This paper has shown a variety of classical techniques for direct calculation of the inductance of magnetic structures. The key to quick calculation is to know which of these techniques to use in a particular situation. By example, the techniques are illustrated for several common magnetic structures.

A detailed bibliography covering inductance calculation techniques may be found at the author's business website at:

http://members.aol.com/marctt/Technical/Inductance_References.htm

The author welcomes comments on this article and additions to the references at marctt@aol.com.

6. REFERENCES

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Marc T. Thompson was born in Vinalhaven Island, Maine. He received the B.S.E.E. degree from the Massachusetts Institute of Technology (MIT) in 1985, the M.S.E.E. in 1992, the Electrical Engineer's degree in 1994, and the Ph.D. in 1997.

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